

Higher-dimensional Contou-Carrère symbol and continuous automorphisms

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Abstract

We prove that the higher-dimensional Contou-Carrère symbol is invariant under continuous automorphisms of algebras of iterated Laurent series over a ring. Applying this property, we obtain a new explicit formula for the higher-dimensional Contou-Carrère symbol. Unlike previously known formulas, this formula is given over an arbitrary ring, not necessarily a \mathbb{Q} -algebra, and does not involve algebraic K -theory.

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1 Introduction

The goal of this paper is to study a relation between the higher-dimensional Contou-Carrère symbol and continuous endomorphisms of the algebra of iterated Laurent series over a ring.

More precisely, given a natural $n \geq 1$, an algebra of iterated Laurent series over a ring A is the algebra $A((t_1)) \dots ((t_n))$, which we denote, for short, by $\mathcal{L}^n(A)$ (see Section 2 for more details). The n -dimensional Contou-Carrère symbol is an antisymmetric multilinear map

$$CC_n : (\mathcal{L}^n(A)^*)^{\times(n+1)} \longrightarrow A^*, \quad (1)$$

which is functorial with respect to a ring A , and where for a ring R we denote by R^* the group of its invertible elements. The one-dimensional Contou-Carrère symbol (or,

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simply, the Contou-Carrère symbol) was first introduced and studied by Contou-Carrère himself, see [2, 3], and by Deligne, see [4]. The two-dimensional Contou-Carrère symbol was introduced and studied by the second named author and Zhu in [11]. For arbitrary n (essentially, for $n > 2$) the higher-dimensional Contou-Carrère symbol was extensively studied by the authors in [7].

In particular, when A is a field, this symbol coincides with the n -dimensional tame symbol from [12], which is a higher-dimensional generalization of the usual tame symbol. When A is a certain Artinian ring over a field k , one obtains from the higher-dimensional Contou-Carrère symbol the higher-dimensional residue. If the field k is finite, then one obtains also the Witt pairing in the higher-dimensional local class field theory, see [11, § 8] and [7, § 9].

The algebra $\mathcal{L}^n(A)$ has a natural topology making it a topological A -module (with the discrete topology on A). For $n = 1$, this topology is the usual topology on $A((t))$ with the base of open neighborhoods of zero given by the A -modules $t^i A[[t]]$, $i \in \mathbb{Z}$. In [8], the authors have studied continuous endomorphisms of A -algebra $\mathcal{L}^n(A)$. In particular, it was obtained an invertibility criterion for such endomorphisms, see [8, Theor. 6.8], together with an explicit formula for the inverse endomorphism, see [8, Rem. 6.4].

Our first main result describes how the higher-dimensional Contou-Carrère symbol is changed under continuous endomorphisms of the A -algebra $\mathcal{L}^n(A)$, see Theorem 3.1. In particular, this implies that the higher-dimensional Contou-Carrère symbol is invariant under continuous automorphisms of the A -algebra $\mathcal{L}^n(A)$, see Corollary 3.2. We provide in Remark 3.3 a generalization of Theorem 3.1 to continuous homomorphisms of A -algebras from $\mathcal{L}^n(A)$ to $\mathcal{L}^m(A)$. Also, we give an example of a non-continuous automorphism of the A -algebra $\mathcal{L}^n(A)$ that does not preserve the higher-dimensional Contou-Carrère symbol, see Proposition 3.4 and Example 3.5.

The invariance result leads to a new explicit formula for the higher-dimensional Contou-Carrère symbol, which is our second main result. Let us explain this in more detail.

Recall that if A is a \mathbb{Q} -algebra, then the map CC_n as in formula (1) is given by an explicit formula, see [7, § 8.4] and also formulas (8), (9), and (10) in Section 2 below. The explicit formula is easily applicable over \mathbb{Q} -algebras. Notice that this formula involves the standard log and exp series, whose coefficients have non-trivial denominators, and thus the formula can not be applied directly over arbitrary rings (in particular, over rings of positive characteristic).

On the other hand, when $n = 1$, for any ring A (not necessarily a \mathbb{Q} -algebra), there is another explicit formula for the Contou-Carrère symbol, see [1, Intr.] and [11, § 2]. This formula is based on the fact that any invertible Laurent series from $A((t))^*$ can be decomposed into an infinite product, where almost all factors are of type $1 + a_l t^l$ with $l \in \mathbb{Z}$, $a_l \in A$.

When $n = 2$, the situation is more delicate. The decomposition of an element from $A((t_1))((t_2))^*$ into an infinite product, where almost all factors are of type $1 + a_{l_1, l_2} t_1^{l_1} t_2^{l_2}$ with $l_1, l_2 \in \mathbb{Z}$, $a_{l_1, l_2} \in A$, was obtained in [11, Prop. 3.14] only when the nil-radical of the ring A is a nilpotent ideal (for example, this holds if A is Noetherian). Correspondingly, an explicit formula for the two-dimensional Contou-Carrère symbol

was obtained in [11] only over such rings, see [11, Def.3.5, Prop.3.16]. Nevertheless, the definition of the two-dimensional Contou-Carrère symbol was given in [11] over an arbitrary ring A by means of generalized commutators in categorical central extensions, whose construction uses, in general, Nisnevich coverings of the scheme $\mathrm{Spec}(A)$.

For arbitrary $n \geq 1$, the definition of the higher-dimensional Contou-Carrère symbol over any ring is given by means of boundary maps for algebraic K -groups, see [11, § 7] and [7, § 8.2].

In this paper, we obtain an explicit formula for the higher-dimensional Contou-Carrère symbol, which is applicable over any ring A and does not use algebraic K -theory, see Definition 4.2, formula (19), Theorem 4.3, and also Remark 4.7. The formula is quite elementary and uses only products of series, the higher-dimensional residue map, and a canonical map $\pi: \mathcal{L}^n(A)^* \rightarrow A^*$, which is induced by a natural decomposition of the group $\mathcal{L}^n(A)^*$, see formula (3) in Section 2. The main idea for the construction of this explicit formula comes from the above invariance property of the higher-dimensional Contou-Carrère symbol combined with an invertibility criterion for continuous endomorphisms of the A -algebra $\mathcal{L}^n(A)$ obtained in [8].

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2 Preliminaries and notation

For short, by a ring, we mean a commutative associative unital ring and similarly for algebras.

Throughout the paper, A denotes a ring and $n \geq 1$ is a positive integer. Let $\mathcal{L}(A) := A((t)) := A[[t]][t^{-1}]$ be the ring of Laurent series over A and let $\mathcal{L}^n(A) := A((t_1)) \dots ((t_n)) = (\mathcal{L}^{n-1}(A))((t_n))$ be the ring of iterated Laurent series over A . Explicitly, an iterated Laurent series $f \in \mathcal{L}^n(A)$ has a form $f = \sum_{l \in \mathbb{Z}^n} a_l t^l$, where $a_l \in A$, $t^l := t_1^{l_1} \dots t_n^{l_n}$ for $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$, and the series satisfies a certain condition on its support, see, e.g., [7, § 3.1].

There is a natural topology on $\mathcal{L}^n(A)$ such that $\mathcal{L}^n(A)$ is a topological group with respect to the addition of iterated Laurent series. The base of open neighborhoods of zero in $\mathcal{L}(A)$ is given by A -submodules $U_i := t^i A[[t]]$, $i \in \mathbb{Z}$. The base of open neighborhoods of zero in $\mathcal{L}^n(A)$ is given by A -submodules

$$U_{i, \{V_j\}} := \left(\bigoplus_{j < i} t_n^j \cdot V_j \right) \oplus t_n^i \cdot \mathcal{L}^{n-1}(A)[[t_n]],$$

where $i \in \mathbb{Z}$ and for each j , $j < i$, the A -module V_j is from the base of open neighborhoods of zero in $\mathcal{L}^{n-1}(A)$. See more details on the topology, e.g., in [7, §§ 3.2, 3.3] and [8, § 2].

Let $L^n \mathbb{G}_m$ be a group functor on the category of rings that sends a ring A to the group of invertible elements $\mathcal{L}^n(A)^*$ in the ring $\mathcal{L}^n(A)$. We have an embedding $\mathbb{G}_m \hookrightarrow L^n \mathbb{G}_m$ given by constant series. Let $\underline{\mathbb{Z}}$ be a group functor on the category of rings that sends a

ring A to the group $\underline{\mathbb{Z}}(A)$ of all \mathbb{Z} -valued locally constant functions on $\text{Spec}(A)$ with the Zariski topology. We have an embedding

$$\underline{\mathbb{Z}}^n \hookrightarrow L^n \mathbb{G}_m, \quad \underline{l} \longmapsto t^{\underline{l}},$$

where $\underline{l} = (l_1, \dots, l_n) \in \underline{\mathbb{Z}}^n(A)$ is a locally constant \mathbb{Z}^n -valued function on $\text{Spec}(A)$. The element $t^{\underline{l}} \in \mathcal{L}^n(A)$ is defined naturally, see details in [7, §4.2]. For example, suppose that $n = 1$ and there is a decomposition into a product of rings $A \simeq B \times B'$ such that $\underline{l} \in \underline{\mathbb{Z}}(A)$ takes the value $m \in \mathbb{Z}$ on $\text{Spec}(B)$ and takes the value $m' \in \mathbb{Z}$ on $\text{Spec}(B')$. Then $t^{\underline{l}}$ is equal to the element $(t^m, t^{m'}) \in \mathcal{L}(B) \times \mathcal{L}(B') \simeq \mathcal{L}(A)$.

Define a lexicographical order on \mathbb{Z}^n such that $(l_1, \dots, l_n) \leq (l'_1, \dots, l'_n)$ if and only if either $l_n < l'_n$, or $l_n = l'_n$ and $(l_1, \dots, l_{n-1}) \leq (l'_1, \dots, l'_{n-1})$. For short, we put $0 := (0, \dots, 0) \in \mathbb{Z}^n$. Let $\mathbb{V}_{n,+}(A)$ be the subgroup of $L^n \mathbb{G}_m(A)$ that consists of all iterated Laurent series of type $1 + \sum_{l > 0} a_l t^l$. Let $\mathbb{V}_{n,-}(A)$ be the subgroup of $L^n \mathbb{G}_m(A)$ that consists of all iterated Laurent series of type $1 + \sum_{l < 0} a_l t^l$ such that $\sum_{l < 0} a_l t^l$ is a nilpotent element of $\mathcal{L}^n(A)$. Then there is the following isomorphism of group functors, see [2], [3] for $n = 1$ and [7, Prop. 4.3] for arbitrary n :

$$L^n \mathbb{G}_m \simeq \underline{\mathbb{Z}}^n \times \mathbb{G}_m \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-}. \quad (2)$$

The decomposition (2) defines morphisms of group functors

$$\nu : L^n \mathbb{G}_m \longrightarrow \underline{\mathbb{Z}}^n, \quad \pi : L^n \mathbb{G}_m \longrightarrow \mathbb{G}_m. \quad (3)$$

In particular, for any i , $1 \leq i \leq n$, we have that $\nu(t_i) = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 stands on the i -th place. The morphism ν is the classical discrete valuation when A is a field and $n = 1$. Let a group functor $(L^n \mathbb{G}_m)^0$ be the kernel of this morphism ν . Clearly, we have a decomposition of group functors

$$L^n \mathbb{G}_m \simeq \underline{\mathbb{Z}}^n \times (L^n \mathbb{G}_m)^0. \quad (4)$$

Suppose for an element $f = \sum_{l \in \mathbb{Z}^n} a_l t^l$ in $\mathcal{L}^n(A)^*$, the function $\nu(f) \in \underline{\mathbb{Z}}^n(A)$ is constant on $\text{Spec}(A)$, that is, we have $\nu(f) = l_0 \in \mathbb{Z}^n$. Then $\pi(f)$ coincides with the coefficient a_{l_0} modulo the nilradical of the ring A , see [7, Lem. 4.7(i)]. However, in general, $\pi(f)$ is not equal to a_{l_0} in A , see [7, Ex. 4.5(iii)].

Let $(L^n \mathbb{G}_m)^\sharp(A)$ consist of all iterated Laurent series $\sum_{l \in \mathbb{Z}^n} a_l t^l \in \mathcal{L}^n(A)$ such that the iterated Laurent series $1 - \sum_{l \leq 0} a_l t^l$ is a nilpotent element of $\mathcal{L}^n(A)$. Then $(L^n \mathbb{G}_m)^\sharp$ is a group subfunctor in $L^n \mathbb{G}_m$ and we have the following decomposition of the group functor $(L^n \mathbb{G}_m)^0$:

$$(L^n \mathbb{G}_m)^0 = \mathbb{G}_m \cdot (L^n \mathbb{G}_m)^\sharp. \quad (5)$$

In addition, we have the following topological characterization of $(L^n \mathbb{G}_m)^\sharp(A)$. For any $f \in \mathcal{L}^n(A)$, one has $f \in (L^n \mathbb{G}_m)^\sharp(A)$ if and only if the sequence $\{(f - 1)^i\}$, $i \in \mathbb{N}$, tends

to zero in $\mathcal{L}^n(A)$, see [7, Def. 3.7, Prop. 3.8]. It follows that if A is a \mathbb{Q} -algebra, then for any $f \in (L^n \mathbb{G}_m)^\#(A)$, the series $\log(f) := \sum_{i \geq 1} (-1)^{i+1} \frac{(f-1)^i}{i}$ converges in $\mathcal{L}^n(A)$, see [7, § 3.3].

Let $\text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$ be the monoid of all continuous endomorphisms of the A -algebra $\mathcal{L}^n(A)$. An endomorphism $\phi \in \text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$ is determined uniquely by the collection $\phi(t_1), \dots, \phi(t_n) \in \mathcal{L}^n(A)^*$. Given a collection $\varphi_1, \dots, \varphi_n \in \mathcal{L}^n(A)^*$, we have an $(n \times n)$ -matrix $(\nu(\varphi_1), \dots, \nu(\varphi_n))$ with entries in $\mathbb{Z}(A)$, where we consider elements $\nu(\varphi_i) \in \mathbb{Z}^n(A)$ as columns. There is an endomorphism $\phi \in \text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$ with $\phi(t_i) = \varphi_i$, $1 \leq i \leq n$, if and only if the matrix $(\nu(\varphi_1), \dots, \nu(\varphi_n))$ is upper-triangular and its diagonal elements are positive point-wise on $\text{Spec}(A)$, see [8, Theor. 4.7].

For an endomorphism $\phi \in \text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$, let $\Upsilon(\phi)$ be the $(n \times n)$ -matrix $(\nu(\phi(t_1)), \dots, \nu(\phi(t_n)))$ with entries in $\mathbb{Z}(A)$. Also, put $d(\phi) := \det(\Upsilon(\phi)) \in \mathbb{Z}(A)$. For any $f \in \mathcal{L}^n(A)^*$, we have the following equality in $\mathbb{Z}^n(A)$, see [8, Prop. 3.10]:

$$\nu(\phi(f)) = \Upsilon(\phi) \cdot \nu(f). \quad (6)$$

An endomorphism ϕ is invertible if and only if the matrix $\Upsilon(\phi)$ is invertible, see [8, Theor. 6.8]. Explicitly, the last condition means that the upper-triangular matrix $\Upsilon(\phi)$ has units on the diagonal, or, equivalently, that $d(\phi) = 1$.

Let $\Omega_{\mathcal{L}^n(A)/A}^1$ be the $\mathcal{L}^n(A)$ -module of Kähler differentials of the ring $\mathcal{L}^n(A)$ over A and put $\Omega_{\mathcal{L}^n(A)/A}^n := \bigwedge_{\mathcal{L}^n(A)}^n \Omega_{\mathcal{L}^n(A)/A}^1$. Let $\tilde{\Omega}_{\mathcal{L}^n(A)}^1$ be the quotient of the $\mathcal{L}^n(A)$ -module $\Omega_{\mathcal{L}^n(A)/A}^1$ by the submodule generated by elements $df - \sum_{i=1}^n \frac{\partial f}{\partial t_i} dt_i$, where $f \in \mathcal{L}^n(A)$. Then $\tilde{\Omega}_{\mathcal{L}^n(A)}^1$ is a free $\mathcal{L}^n(A)$ -module of rank n . Put $\tilde{\Omega}_{\mathcal{L}^n(A)}^n := \bigwedge_{\mathcal{L}^n(A)}^n \tilde{\Omega}_{\mathcal{L}^n(A)}^1$. One has an A -linear residue map

$$\text{res} : \tilde{\Omega}_{\mathcal{L}^n(A)}^n \longrightarrow A, \quad \sum_{l \in \mathbb{Z}^n} a_l t^l \cdot dt_1 \wedge \dots \wedge dt_n \longmapsto a_{-1 \dots -1}.$$

We denote the natural composition

$$\Omega_{\mathcal{L}^n(A)/A}^n \longrightarrow \tilde{\Omega}_{\mathcal{L}^n(A)}^n \xrightarrow{\text{res}} A$$

also by res .

Any endomorphism $\phi \in \text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$ defines naturally an A -linear endomorphism of $\tilde{\Omega}_{\mathcal{L}^n(A)}^n$, which we denote also by ϕ for simplicity. We have the following equality, see [8, Cor. 5.4]:

$$\text{res}(\phi(\omega)) = d(\phi) \text{res}(\omega). \quad (7)$$

In [7] the authors have introduced and studied a special class of ind-affine schemes called thick ind-cones, see [7, § 5.4] for more details. The point is that many functors that occur in the context of iterated Laurent series are represented by ind-affine schemes which are products of ind-flat ind-affine schemes over \mathbb{Z} and thick ind-cones (in the one-dimensional case, the corresponding ind-affine schemes are just ind-flat over \mathbb{Z}).

Such products possess many nice properties, mainly concerning regular functions on them. In particular, if an ind-affine scheme X is a product of an ind-flat ind-affine scheme over \mathbb{Z} and a thick ind-cone, then the natural homomorphism between algebras of regular functions $\mathcal{O}(X) \rightarrow \mathcal{O}(X_{\mathbb{Q}})$ is injective, see [7, Prop. 5.17]. Besides, for all such ind-schemes X and Y , their product $X \times Y$ is also isomorphic to a product of an ind-flat ind-affine scheme over \mathbb{Z} and a thick ind-cone, see [7, Lem. 5.13].

The functors $L^n \mathbb{G}_m$, $(L^n \mathbb{G}_m)^0$, $(L^n \mathbb{G}_m)^\sharp$, $\mathbb{V}_{n,+}$, and $\mathbb{V}_{n,-}$ are represented by ind-affine schemes which are products of ind-flat ind-affine schemes over \mathbb{Z} and thick ind-cones, see [7, § 6.3].

The assignment

$$A \longmapsto \text{End}_A^{\text{c,alg}}(\mathcal{L}^n(A))$$

is a functor, which we denote by $\mathcal{E}nd^{\text{c,alg}}(\mathcal{L}^n)$, see [8, § 5]. This functor is also represented by an ind-affine scheme which is a product of an ind-flat ind-affine scheme over \mathbb{Z} and a thick ind-cone, see [8, Prop. 5.1].

Given a functor on the category of rings which is represented by an ind-scheme, we use the same notation for the representing ind-scheme as for the initial functor.

By [7, Prop. 8.15], there is a unique multilinear (anti)symmetric map

$$\text{sgn} : (\mathbb{Z}^n)^{\times(n+1)} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

such that for all $l_1, \dots, l_n \in \mathbb{Z}^n$, we have $\text{sgn}(l_1, l_1, l_2, \dots, l_n) \equiv \det(l_1, l_2, \dots, l_n) \pmod{2}$. It follows that the map sgn is invariant under automorphisms of \mathbb{Z}^n . One has (equivalent) explicit formulas for the map sgn , see the proof of [7, Prop. 8.15] and [7, Rem. 8.16].

By [7, Prop. 8.22, Rem. 8.21] (see also [5]), for any \mathbb{Q} -algebra A , there is a unique multilinear antisymmetric map

$$CC_n : (\mathcal{L}^n(A)^*)^{\times(n+1)} \longrightarrow A^*$$

that satisfies the following properties:

(i) if $f_1 \in (L^n \mathbb{G}_m)^\sharp(A)$, then

$$CC_n(f_1, f_2, \dots, f_{n+1}) = \exp \text{ res} \left(\log(f_1) \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_{n+1}}{f_{n+1}} \right), \quad (8)$$

where the standard series \exp is applied to a nilpotent element of the ring A ;

(ii) if $f_1 \in A^*$, then

$$CC_n(f_1, f_2, \dots, f_{n+1}) = f_1^{\det(\nu(f_2), \dots, \nu(f_{n+1}))}, \quad (9)$$

(iii) for all $l_1, \dots, l_{n+1} \in \mathbb{Z}(A)$, there is an equality

$$CC_n(t^{l_1}, \dots, t^{l_{n+1}}) = (-1)^{\text{sgn}(l_1, \dots, l_{n+1})}. \quad (10)$$

It is not not hard to deduce from this definition of CC_n that for all elements $f_1, \dots, f_n \in \mathcal{L}^n(A)^*$, there is an equality

$$CC_n(f_1, f_1, f_2, \dots, f_n) = (-1)^{\det(\nu(f_1), \dots, \nu(f_n))}. \quad (11)$$

Indeed, by [7, Lem. 8.24], we have that $CC_n(f_1, f_1, f_2, \dots, f_n) = CC_n(-1, f_1, \dots, f_n)$ and then we apply formula (9).

Clearly, the map CC_n is functorial with respect to a \mathbb{Q} -algebra A , that is, we have a morphism of functors $CC_n: (L^n \mathbb{G}_m)_{\mathbb{Q}}^{\times(n+1)} \rightarrow (\mathbb{G}_m)_{\mathbb{Q}}$ on the category of \mathbb{Q} -algebras. Since the functor $L^n \mathbb{G}_m$ is represented by a product of an ind-flat ind-affine scheme over \mathbb{Z} and a thick ind-cone, the above discussion on such ind-schemes implies that there is at most one extension of the above map CC_n to a morphism of functors $(L^n \mathbb{G}_m)^{\times(n+1)} \rightarrow \mathbb{G}_m$ on the category of all rings. In addition, this extension must be automatically multilinear, antisymmetric and must satisfy formula (11), which also follows from the theory of thick ind-cones.

Actually, such extension does exist. Namely, based on the works of Grayson [9] and Kato [10], one constructs a boundary map between algebraic K -groups

$$\partial_{m+1} : K_{m+1}(\mathcal{L}(A)) \longrightarrow K_m(A), \quad m \geq 0,$$

which is functorial with respect to a ring A , see [11, § 7] and [7, § 7.3]. Recall that for any ring B , there is a surjective homomorphism $\det: K_1(B) \rightarrow B^*$, which has a section given by a canonical embedding $B^* \subset K_1(B)$. This allows to define the composition

$$\begin{aligned} (\mathcal{L}^n(A)^*)^{\times(n+1)} &\longrightarrow K_1(\mathcal{L}^n(A))^{\times(n+1)} \longrightarrow \\ &\longrightarrow K_{n+1}(\mathcal{L}^n(A)) \xrightarrow{\partial_2 \dots \partial_{n+1}} K_1(A) \xrightarrow{\det} A^*, \end{aligned} \quad (12)$$

where the second map is the product between algebraic K -groups.

A non-trivial theorem is that if A is a \mathbb{Q} -algebra, then the composition in formula (12) coincides with the map defined above explicitly by formulas (8), (9), and (10), see [7, Theor. 8.17]. Thus we denote also by CC_n the morphism of functors $(L^n \mathbb{G}_m)^{\times(n+1)} \rightarrow \mathbb{G}_m$ obtained from formula (12) and call it a higher-dimensional Contou-Carrère symbol.

3 Invariance of the higher-dimensional Contou-Carrère symbol

Theorem 3.1. *Let $\phi: \mathcal{L}^n(A) \rightarrow \mathcal{L}^n(A)$ be a continuous endomorphism of the A -algebra $\mathcal{L}^n(A)$. Then for all elements $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, there is an equality in A^**

$$CC_n(\phi(f_1), \dots, \phi(f_{n+1})) = CC_n(f_1, \dots, f_{n+1})^{d(\phi)}. \quad (13)$$

Proof. Both sides of formula (13) are regular functions on the ind-affine scheme $X := \mathcal{E}nd^{c, \text{alg}}(\mathcal{L}^n) \times (L^n \mathbb{G}_m)^{\times(n+1)}$. The scheme X is isomorphic to a product of an ind-flat ind-affine scheme over \mathbb{Z} and a thick ind-cone. Therefore, by results mentioned in Section 2, the natural homomorphism between algebras of regular functions $\mathcal{O}(X) \rightarrow \mathcal{O}(X_{\mathbb{Q}})$

is injective. Hence it is enough to prove formula (13) when A is a \mathbb{Q} -algebra, which we assume from now on.

Further, both sides of formula (13) are multilinear and antisymmetric with respect to f_1, \dots, f_{n+1} . Thus decompositions (4) and (5) imply that it is enough to prove formula (13) in the following three cases:

- (i) we have that $f_1 \in (L^n \mathbb{G}_m)^\sharp(A)$;
- (ii) we have that $f_1 \in A^*$;
- (iii) there is a collection $\underline{l}_1, \dots, \underline{l}_{n+1} \in \underline{\mathbb{Z}}^n(A)$ such that $f_i = t^{\underline{l}_i}$, where $1 \leq i \leq n+1$.

Suppose that condition (i) holds. Since ϕ is continuous, the topological characterization of the group $(L^n \mathbb{G}_m)^\sharp(A)$ given in Section 2 implies that ϕ preserves $(L^n \mathbb{G}_m)^\sharp(A)$, whence $\phi(f_1) \in (L^n \mathbb{G}_m)^\sharp(A)$. Consequently, by formulas (8) and (7), the left hand side of (13) is equal to

$$\begin{aligned} \exp \operatorname{res} \left(\log(\phi(f_1)) \frac{d\phi(f_2)}{\phi(f_2)} \wedge \dots \wedge \frac{d\phi(f_{n+1})}{\phi(f_{n+1})} \right) &= \exp \operatorname{res} \left(\phi \left(\log(f_1) \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right) = \\ &= \exp \left(d(\phi) \operatorname{res} \left(\log(f_1) \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right) = \exp \left(\operatorname{res} \left(\log(f_1) \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right)^{d(\phi)}. \end{aligned}$$

Again by formula (8), this equals to the right hand side of (13).

Now suppose that condition (ii) holds. Since ϕ is A -linear, we have that $\phi(f_1) = f_1$. By [7, Prop. 8.4], there are equalities

$$\begin{aligned} \det(\nu(f_2), \dots, \nu(f_{n+1})) &= \operatorname{res} \left(\frac{df_2}{f_2} \wedge \dots \wedge \frac{df_{n+1}}{f_{n+1}} \right), \\ \det(\nu(\phi(f_2)), \dots, \nu(\phi(f_{n+1}))) &= \operatorname{res} \left(\frac{d\phi(f_2)}{\phi(f_2)} \wedge \dots \wedge \frac{d\phi(f_{n+1})}{\phi(f_{n+1})} \right). \end{aligned}$$

By formulas (7) and (9), this proves (13) when $f_1 \in A^*$.

Finally, suppose that condition (iii) holds. There is a decomposition into a finite product of rings

$$A \simeq \prod_{j=1}^N A_j \tag{14}$$

such that for all i and j with $1 \leq i \leq n+1$, $1 \leq j \leq N$, the restriction of the function \underline{l}_i to $\operatorname{Spec}(A_j)$ is constant. Since both sides of formula (13) are functorial with respect to the terms of decomposition (14) (actually, with respect to any A -algebra), it is enough to prove formula (13) for each ring A_j separately. Thus we may assume that all \underline{l}_i are constant, that is, are elements of $\mathbb{Z}^n \subset \underline{\mathbb{Z}}^n(A)$.

It follows from multilinearity and the antisymmetric property that it is enough to consider the case when the collection (f_1, \dots, f_{n+1}) is of type $(t_{p_1}, t_{p_1}, t_{p_2}, \dots, t_{p_n})$ for some $1 \leq p_1, \dots, p_n \leq n$. By [7, Lem. 8.24], the values of both sides of formula (13) at such collection coincide with their values at the collection $(-1, t_{p_1}, t_{p_2}, \dots, t_{p_n})$. Thus we are reduced to the case (ii) considered above.

□

Corollary 3.2. *If ϕ is a continuous automorphism of the A -algebra $\mathcal{L}^n(A)$, then for all elements $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, there is an equality in A^**

$$CC_n(\phi(f_1), \dots, \phi(f_{n+1})) = CC_n(f_1, \dots, f_{n+1}).$$

Proof. Since the endomorphism ϕ is invertible, we have that $d(\phi) = 1$, see Section 2. Thus the corollary follows from Theorem 3.1. \square

Remark 3.3. Theorem 3.1 has the following generalization. Let $\phi: \mathcal{L}^n(A) \rightarrow \mathcal{L}^m(A)$ be a continuous homomorphism of A -algebras. In particular, by [8, Cor. 4.8], we have that $n \leq m$. One defines invariants $\text{sgn}(\phi)$ and $d(\phi)$ in $\mathbb{Z}(A)$, see [8, § 5] (when $m = n$, we have that $\text{sgn}(\phi) = 1$ and $d(\phi)$ is $\det(\Upsilon(\phi))$ as above). The homomorphism ϕ defines also a collection q_1, \dots, q_{m-n} of elements from $\mathbb{Z}(A)$ such that for any form $\omega \in \tilde{\Omega}_{\mathcal{L}^n(A)}^n$, one has the following equality, see [8, Prop. 5.3]:

$$\text{res} \left(\phi(\omega) \wedge \frac{dt_{q_1}}{t_{q_1}} \wedge \dots \wedge \frac{dt_{q_{m-n}}}{t_{q_{m-n}}} \right) = \text{sgn}(\phi) d(\phi) \text{res}(\omega).$$

A similar argument as in the proof of Theorem 3.1 shows that for all elements $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, there is an equality in A^*

$$CC_m(\phi(f_1), \dots, \phi(f_{n+1}), t_{q_1}, \dots, t_{q_{m-n}}) = CC_n(f_1, \dots, f_{n+1})^{\text{sgn}(\phi)d(\phi)}.$$

Now let us show that if an automorphism of the A -algebra $\mathcal{L}^n(A)$ is not continuous, then, in general, it does not preserve the higher-dimensional Contou-Carrère symbol.

Let ϕ be an automorphism of the A -algebra $\mathcal{L}^n(A)$ (not necessarily continuous). Then ϕ defines naturally an A -linear automorphism of $\Omega_{\mathcal{L}^n(A)/A}^n$, which we denote also by ϕ for simplicity. Let ε be a formal variable that satisfies $\varepsilon^2 = 0$. By ϕ_ε denote the induced automorphism of the $A[\varepsilon]$ -algebra $\mathcal{L}^n(A[\varepsilon]) \simeq \mathcal{L}^n(A)[\varepsilon]$.

Proposition 3.4. *Assume that there is a differential form $\omega \in \Omega_{\mathcal{L}^n(A)/A}^n$ such that*

$$\text{res}(\phi(\omega)) \neq \text{res}(\omega).$$

Then there is a collection $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A[\varepsilon])^$ such that*

$$CC_n(\phi_\varepsilon(f_1), \dots, \phi_\varepsilon(f_{n+1})) \neq CC_n(f_1, \dots, f_{n+1}).$$

Proof. Suppose that ϕ_ε preserves the higher-dimensional Contou-Carrère symbol. By [7, Prop. 8.19], for any collection $f_1, \dots, f_n \in \mathcal{L}^n(A)^*$ and any element $g \in \mathcal{L}^n(A)$, there is an equality

$$CC_n(1 + g\varepsilon, f_1, \dots, f_n) = 1 + \text{res} \left(g \frac{df_1}{f_1} \wedge \dots \wedge \frac{df_n}{f_n} \right) \varepsilon, \quad (15)$$

where CC_n is applied to a collection of invertible elements of the ring $\mathcal{L}^n(A[\varepsilon])$. Thus we see that ϕ preserves the residue in the right hand side of formula (15).

Note that $\mathcal{L}^n(A)$ is additively generated by invertible elements, see [6, Ex. 2.3(iii)]. It follows that the additive group $\Omega_{\mathcal{L}^n(A)/A}^n$ is generated by differential forms as in the right hand side of formula (15). Hence ϕ preserves the residue of all differential forms in $\Omega_{\mathcal{L}^n(A)/A}^n$, which contradicts the assumption of the proposition. \square

Example 3.5. Yekutieli has given in [13, Ex. 2.4.24] the following construction of a non-continuous automorphism ϕ of the field $k((t_1))((t_2))$ over a field k of zero characteristic such that ϕ does not preserve the residue. Let $f \in k((t_1))$ be an element which is transcendental over the subfield $k(t_1) \subset k((t_1))$. It follows from Hensel's lemma that there is a (non-unique) section $\phi_0: k((t_1)) \rightarrow k((t_1))[[t_2]]$ of the natural homomorphism $k((t_1))[[t_2]] \rightarrow k((t_1))$ such that ϕ_0 is identical on $k(t_1)$ and $\phi_0(f) = f + t_2$. One shows directly that ϕ_0 extends to an automorphism of the ring $k((t_1))[[t_2]]$ that sends t_2 to itself. Thus we obtain an automorphism ϕ of the field $k((t_1))((t_2))$ over k such that

$$\phi(t_1) = t_1, \quad \phi(t_2) = t_2, \quad \phi(f) = f + t_2.$$

It follows that

$$\phi \left(\frac{dt_1}{t_1} \wedge \frac{df}{t_2} \right) = \frac{dt_1}{t_1} \wedge \frac{df}{t_2} + \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}.$$

Since $f \in k((t_1))$, the image of $\frac{dt_1}{t_1} \wedge \frac{df}{t_2}$ under the natural map $\Omega_{k((t_1))((t_2))/k}^2 \rightarrow \tilde{\Omega}_{k((t_1))((t_2))}^2$ equals zero. Hence we have

$$\text{res} \left(\frac{dt_1}{t_1} \wedge \frac{df}{t_2} \right) = 0, \quad \text{res} \phi \left(\frac{dt_1}{t_1} \wedge \frac{df}{t_2} \right) = 1.$$

Therefore by our Proposition 3.4, the automorphism ϕ_ε does not preserve the two-dimensional Contou-Carrère symbol over the ring $k[\varepsilon]((t_1))((t_2))$. Explicitly, we have that

$$CC_2 \left(1 + \phi \left(\frac{f}{t_2} \right) \varepsilon, \phi(t_1), \phi(f) \right) \neq CC_2 \left(1 + \frac{f}{t_2} \varepsilon, t_1, f \right),$$

which follows from formula (15).

4 A new formula for the higher-dimensional Contou-Carrère symbol

We will use the following simple fact. Let L be a free \mathbb{Z} -module of rank $2n$ with a basis $\{e_1, \dots, e_n, x_1, \dots, x_n\}$. Let \mathcal{K} be the set of all pairs (K, κ) , where K is a (possibly, empty) subset of $\{1, \dots, n\}$ and $\kappa: K \hookrightarrow \{1, \dots, n\}$ is an order-preserving injective map (which might differ from the initial embedding). For each $(K, \kappa) \in \mathcal{K}$, define an element

$$v_{(K, \kappa)} := v_1 \wedge \dots \wedge v_n \in \Lambda^n L, \tag{16}$$

where for each i , $1 \leq i \leq n$, we put

$$v_i := \begin{cases} e_i, & \text{if } i \notin K, \\ e_i + x_{\kappa(i)}, & \text{if } i \in K. \end{cases}$$

For example, we have that

$$v_\emptyset = e_1 \wedge \dots \wedge e_n, \quad v_{(\{1, \dots, n\}, \text{id})} = (e_1 + x_1) \wedge \dots \wedge (e_n + x_n). \tag{17}$$

Lemma 4.1. *The set $\{v_{(K,\kappa)}\}$, where $(K,\kappa) \in \mathcal{K}$, is a basis of the \mathbb{Z} -module $\Lambda^n L$.*

Proof. We claim that the set \mathcal{K} is naturally bijective with the set of n -combinations from a set of $2n$ elements. Indeed, a pair (K,κ) corresponds to the subset

$$K \sqcup \overline{\kappa(K)} \subset \{1, \dots, n\} \sqcup \{1, \dots, n\} \simeq \{1, \dots, 2n\},$$

where $\overline{\kappa(K)} := \{1, \dots, n\} \setminus \kappa(K)$. Hence the set \mathcal{K} has $\binom{2n}{n}$ elements, which is also equal to the rank of $\Lambda^n L$ over \mathbb{Z} .

We see that it is enough to show that the set $\{v_{(K,\kappa)}\}$, where $(K,\kappa) \in \mathcal{K}$, generates the \mathbb{Z} -module $\Lambda^n L$. Consider an element $e_{p_1} \wedge \dots \wedge e_{p_m} \wedge x_{r_1} \wedge \dots \wedge x_{r_{n-m}} \in \Lambda^n L$, where $0 \leq m \leq n$, $1 \leq p_1 < \dots < p_m \leq n$, and $1 \leq r_1 < \dots < r_{n-m} \leq n$ (in particular, when $m = 0$, there is no p_i and when $m = n$, there is no r_j). Let $1 \leq q_1 < \dots < q_{n-m} \leq n$ be the complementary collection to $\{p_1, \dots, p_m\}$ in $\{1, \dots, n\}$. Let σ be the permutation that sends $(1, \dots, n)$ to $(p_1, \dots, p_m, q_1, \dots, q_{n-m})$. Define an element $(M, \mu) \in \mathcal{K}$, where $M := \{q_1, \dots, q_{n-m}\}$ and $\mu(q_i) := r_i$, where $1 \leq i \leq n - m$.

Then there is an equality in $\Lambda^n L$

$$e_{p_1} \wedge \dots \wedge e_{p_m} \wedge x_{r_1} \wedge \dots \wedge x_{r_{n-m}} = (-1)^{\text{sgn}(\sigma) + n - m} \sum_{K \subset M} (-1)^{|K|} v_{(K,\kappa)}, \quad (18)$$

where K runs over all subsets of M (including the empty set), κ is the restriction of μ to K , and $|K|$ denotes the number of elements in the set K . This is obtained directly as a result of opening brackets after we replace x_{r_i} by the equal expression $(e_{q_i} + x_{r_i}) - e_{q_i}$ for all i , $1 \leq i \leq n - m$.

Since the elements $e_{p_1} \wedge \dots \wedge e_{p_m} \wedge x_{r_1} \wedge \dots \wedge x_{r_{n-m}}$ generate $\Lambda^n L$, this finishes the proof. \square

Let $g = (g_1, \dots, g_n)$ be a collection of invertible iterated Laurent series from $\mathcal{L}^n(A)^*$ such that the $(n \times n)$ -matrix $(\nu(g)) := (\nu(g_1), \dots, \nu(g_n))$ is upper-triangular with units on the diagonal. As it was explained in Section 2, there is a unique continuous automorphism of the A -algebra $\phi_g: \mathcal{L}^n(A) \rightarrow \mathcal{L}^n(A)$ such that $\phi_g(t_i) = g_i$ for all i , $1 \leq i \leq n$. For any element $f \in \mathcal{L}^n(A)^*$, put

$$\langle f, g \rangle = \langle f, g_1, \dots, g_n \rangle := \pi(\phi_g^{-1}(f)),$$

where π is defined in Section 2, see formula (3). In particular, we have the equality $\langle f, t_1, \dots, t_n \rangle = \pi(f)$.

Recall that in [8, Rem. 6.4, Theor. 6.8], it is given an explicit formula for the inverse to a continuous automorphism of the A -algebra $\mathcal{L}^n(A)$. This implies the equality

$$\langle f, g \rangle = \pi \left(\sum_{l \in \mathbb{Z}^n} \text{res} (f g^{-l-1} J(g) dt_1 \wedge \dots \wedge dt_n) t^l \right), \quad (19)$$

where $g^{-l-1} := g_1^{-l_1-1} \dots g_n^{-l_n-1}$ and $J(g) \in \mathcal{L}(A)$ is the Jacobian of ϕ_g , that is, the determinant of the matrix $\left(\frac{\partial g_i}{\partial t_j} \right)$.

Let $e := (e_1, \dots, e_n)$ be a row of elements in the \mathbb{Z} -module L . For any element $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$ considered as a column, it is defined an element $e \cdot l = \sum_{i=1}^n l_i e_i \in L$.

Clearly, for any ring A , one generalizes the above notation and facts (in particular, Lemma 4.1) by means of replacement of L by a free $\mathbb{Z}(A)$ -module L of rank $2n$ with a basis $\{e_1, \dots, e_n, x_1, \dots, x_n\}$, where we denoted this module also by L for short.

Definition 4.2. For any ring A and any collection $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, define an element of A^*

$$\widetilde{CC}_n(f_1, \dots, f_{n+1}) := (-1)^{\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1}))} \prod_{(K, \kappa) \in \mathcal{K}} \langle f_1, g_{(K, \kappa)} \rangle^{C(K, \kappa)}, \quad (20)$$

where $C(K, \kappa)$ are elements of $\mathbb{Z}(A)$ such that there is an equality in the free $\mathbb{Z}(A)$ -module $\Lambda^n L$

$$(e \cdot \nu(f_2) + x_1) \wedge \dots \wedge (e \cdot \nu(f_{n+1}) + x_n) = \sum_{(K, \kappa) \in \mathcal{K}} C(K, \kappa) v_{(K, \kappa)}$$

with $v_{(K, \kappa)}$ being defined as in (16), and for each $(K, \kappa) \in \mathcal{K}$, we put

$$g_{(K, \kappa)} := (g_1, \dots, g_n)$$

with

$$g_i := \begin{cases} t_i, & \text{if } i \notin K, \\ t_i \cdot f_{\kappa(i)+1} \cdot t^{-\nu(f_{\kappa(i)+1})}, & \text{if } i \in K \end{cases}$$

for each i , $1 \leq i \leq n$.

Note that the elements $C(K, \kappa) \in \mathbb{Z}(A)$ are well-defined by Lemma 4.1. Also, one checks easily that for any $(K, \kappa) \in \mathcal{K}$, the $(n \times n)$ -matrix $(\nu(g_{(K, \kappa)}))$ is the identity. Hence $\langle f, g_{(K, \kappa)} \rangle$ is a well-defined element of A^* .

For example, we have that (cf. formula (17))

$$g_\emptyset = (t_1, \dots, t_n), \quad g_{(\{1, \dots, n\}, \text{id})} = (t_1 \cdot f_2 \cdot t^{-\nu(f_2)}, \dots, t_n \cdot f_{n+1} \cdot t^{-\nu(f_{n+1})}). \quad (21)$$

Clearly, the map \widetilde{CC}_n is functorial with respect to a ring A .

Theorem 4.3. For any ring A and any collection $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, there is an equality

$$\widetilde{CC}_n(f_1, \dots, f_{n+1}) = CC_n(f_1, \dots, f_{n+1}),$$

where the map CC_n is defined by formula (12).

Proof. As mentioned in Section 2, the morphism of functors $(L^n \mathbb{G}_m)^{\times(n+1)} \rightarrow \mathbb{G}_m$ defined by formula (12) coincides after restriction to \mathbb{Q} -algebras with the morphism of functors $(L^n \mathbb{G}_m)_{\mathbb{Q}}^{\times(n+1)} \rightarrow (\mathbb{G}_m)_{\mathbb{Q}}$ defined by formulas (8), (9), and (10). Moreover, there is only

one morphism of functors $(L^n \mathbb{G}_m)^{\times(n+1)} \rightarrow \mathbb{G}_m$ with this property, that is, which coincides with the given one after restriction to \mathbb{Q} -algebras. Indeed, the theory of thick ind-cones implies that the natural homomorphism $\mathcal{O}((L^n \mathbb{G}_m)^{\times(n+1)}) \rightarrow \mathcal{O}((L^n \mathbb{G}_m)_{\mathbb{Q}}^{\times(n+1)})$ is injective. Thus we may assume that A is a \mathbb{Q} -algebra, which we do from now on.

We will use the following auxiliary multilinear antisymmetric map: for any collection $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, put

$$F(f_1, \dots, f_{n+1}) := (-1)^{\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1}))} CC_n(f_1, \dots, f_{n+1}) \in A^*.$$

We claim that the map $F: (\mathcal{L}^n(A)^*)^{\times(n+1)} \rightarrow A^*$ is also alternating, that is, for any collection $f_1, \dots, f_n \in \mathcal{L}^n(A)^*$, there is an equality $F(f_1, f_1, f_2, \dots, f_n) = 1$. Indeed, by formula (11), there is an equality

$$F(f_1, f_1, f_2, \dots, f_n) = (-1)^{\text{sgn}(\nu(f_1), \nu(f_1), \dots, \nu(f_n))} (-1)^{\det(\nu(f_1), \dots, \nu(f_n))}.$$

Further, by the properties of the map sgn from Section 2, the right hand side of the latter formula is equal to

$$(-1)^{\det(\nu(f_1), \dots, \nu(f_n))} (-1)^{\det(\nu(f_1), \dots, \nu(f_n))} = 1.$$

For any $f \in \mathcal{L}^n(A)^*$ and $\underline{l} \in \underline{\mathbb{Z}}(A)$, the assignment $f \mapsto f^{\underline{l}}$ gives the structure of a $\underline{\mathbb{Z}}(A)$ -module on the group $\mathcal{L}^n(A)^*$. Now consider a collection $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$. Define a homomorphism of $\underline{\mathbb{Z}}(A)$ -modules

$$\alpha : L \longrightarrow \mathcal{L}^n(A)^*, \quad e_i \longmapsto t_i, \quad x_i \longmapsto f_{i+1} \cdot t^{-\nu(f_{i+1})}, \quad (22)$$

where $1 \leq i \leq n$. Since the map F is multilinear and alternating, this defines a homomorphism of $\underline{\mathbb{Z}}(A)$ -modules

$$\beta : \Lambda^n L \longrightarrow A^*, \quad u_1 \wedge \dots \wedge u_n \longmapsto F(f_1, \alpha(u_1), \dots, \alpha(u_n)),$$

where $u_1, \dots, u_n \in L$. By construction, we have the equalities in A^*

$$\beta((e \cdot \nu(f_2) + x_1) \wedge \dots \wedge (e \cdot \nu(f_{n+1}) + x_n)) = F(f_1, f_2, \dots, f_{n+1}),$$

$$\beta(v_{(K, \kappa)}) = F(f_1, g_{(K, \kappa)})$$

for any $(K, \kappa) \in \mathcal{K}$. This implies that there is an equality

$$F(f_1, \dots, f_{n+1}) = \prod_{(K, \kappa) \in \mathcal{K}} F(f_1, g_{(K, \kappa)})^{C(K, \kappa)}.$$

As mentioned in Section 2, the map sgn is invariant under automorphisms of the $\underline{\mathbb{Z}}(A)$ -module $\underline{\mathbb{Z}}^n(A)$. Together with formula (6) and Corollary 3.2 this implies that F is invariant under continuous automorphisms of the A -algebra $\mathcal{L}^n(A)$. Thus for any $(K, \kappa) \in \mathcal{K}$, there is an equality

$$F(f_1, g_{(K, \kappa)}) = F(\phi_{g_{(K, \kappa)}}^{-1}(f_1), t_1, \dots, t_n).$$

Therefore it remains to prove that for any element $f \in \mathcal{L}^n(A)^*$, there is an equality

$$F(f, t_1, \dots, t_n) = \pi(f). \quad (23)$$

If $f \in A^*$, then by formula (9) both sides of the formula (23) are equal to f itself. If $f = t^l$ for some $l \in \mathbb{Z}^n(A)$, or $f \in \mathbb{V}_{n,+}$, or $f \in \mathbb{V}_{n,-}$, then by formula (10) or (8), respectively, both sides of formula (23) are equal to 1 (recall we are assuming that A is a \mathbb{Q} -algebra). Now formula (23) follows from decomposition (2). This finishes that proof of the theorem. \square

Remark 4.4. For some particular choices of elements $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$, there are other formulas of the same type as formula (20) which also give the higher-dimensional Contou-Carrère symbol (cf. Example 4.5(i),(iii) below). Indeed, it follows from the proof of Theorem 4.3 that one has the equality

$$CC_n(f_1, \dots, f_{n+1}) = (-1)^{\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1}))} \prod_{(K, \kappa) \in \mathcal{K}} \langle f_1, g_{(K, \kappa)} \rangle^{D(K, \kappa)},$$

where

$$u_1 \wedge \dots \wedge u_n = \sum_{(K, \kappa) \in \mathcal{K}} D(K, \kappa) v_{(K, \kappa)} \in \Lambda^n L$$

and $u_1, \dots, u_n \in L$ are any elements that satisfy the condition $\alpha(u_i) = \alpha(e \cdot \nu(f_{i+1}) + x_i)$ for all i , $1 \leq i \leq n$, with the map $\alpha: L \rightarrow \mathcal{L}^n(A)^*$ defined by formula (22).

Example 4.5. Here are corollaries of Theorem 4.3 combined with formula (20) and Remark 4.4.

(i) If $n = 1$, then for all elements $f_1, f_2 \in \mathcal{L}(A)^* = A((t))^*$, there is an equality

$$\begin{aligned} CC_1(f_1, f_2) &= (-1)^{\nu(f_1)\nu(f_2)} \langle f_1, t \rangle^{\nu(f_2)-1} \langle f_1, t^{1-\nu(f_2)} \cdot f_2 \rangle = \\ &= (-1)^{\nu(f_1)\nu(f_2)} \pi(f_1)^{\nu(f_2)-1} \langle f_1, t^{1-\nu(f_2)} \cdot f_2 \rangle. \end{aligned} \quad (24)$$

Indeed, in this case, $\text{sgn}(\nu(f_1), \nu(f_2)) \equiv \nu(f_1)\nu(f_2) \pmod{2}$, the set \mathcal{K} consists of two elements: the empty set \emptyset and $(\{1\}, \text{id})$, and by formula (17), we have

$$e \cdot \nu(f_2) + x = (\nu(f_2) - 1)v_{\emptyset} + v_{(\{1\}, \text{id})}.$$

Now we apply formula (21).

Note that if $f_2 = t^l$ for some $l \in \mathbb{Z}(A)$, then the right hand side of formula (24) takes the form

$$(-1)^{\nu(f_1)\nu(f_2)} \pi(f_1)^{\nu(f_2)-1} \langle f_1, t \rangle = (-1)^{\nu(f_1)\nu(f_2)} \pi(f_1)^{\nu(f_2)}.$$

This agrees with Remark 4.4, because in this case, there is an equality $\alpha(e \cdot \nu(f_2) + x) = \alpha(e \cdot \nu(f_2))$ and we have $e \cdot \nu(f_2) = \nu(f_2)v_{\emptyset}$.

(ii) If the $(n \times n)$ -matrix $(\nu(f_2), \dots, \nu(f_{n+1}))$ is the identity, then there is an equality

$$CC_n(f_1, f_2, \dots, f_{n+1}) = (-1)^{\sum_{i=1}^n l_i} \langle f_1, f_2, \dots, f_{n+1} \rangle,$$

where $\nu(f_1) = (l_1, \dots, l_n)$, $l_i \in \mathbb{Z}(A)$. Indeed, in this case, we have $\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1})) \equiv \sum_{i=1}^n l_i \pmod{2}$ (it is enough to check this when $\nu(f_1)$ is an element of the standard basis in $\mathbb{Z}(A)$ -module $\mathbb{Z}^n(A)$, in which case this is obvious). Further, by formula (17), we have

$$(e \cdot \nu(f_2) + x_1) \wedge \dots \wedge (e \cdot \nu(f_{n+1}) + x_n) = v_{\{1, \dots, n\}, \text{id}}$$

and we apply formula (21).

(iii) Suppose that there is a collection $1 \leq p_1 < \dots < p_m \leq n$ with $0 \leq m \leq n$ such that $f_2 = t_{p_1}, \dots, f_{m+1} = t_{p_m}$ and $\nu(f_{m+2}) = \dots = \nu(f_{n+1}) = 0$. Let $1 \leq q_1 < \dots < q_{n-m} \leq n$ be the complementary collection to $\{p_1, \dots, p_m\}$ in $\{1, \dots, n\}$. Let σ be the permutation that sends $(1, \dots, n)$ to $(p_1, \dots, p_m, q_1, \dots, q_{n-m})$. Define an element $(M, \mu) \in \mathcal{K}$, where $M := \{q_1, \dots, q_{n-m}\}$ and $\mu(q_i) := m + i$, $1 \leq i \leq n - m$. Then there is an equality

$$CC_n(f_1, f_2, \dots, f_{n+1}) = (-1)^{\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1}))} \prod_{K \subset M} \langle f_1, g_{(K, \kappa)} \rangle^{(-1)^{(|K| + \text{sgn}(\sigma) + n - m)}}, \quad (25)$$

where K runs over all subsets of M (including the empty set) and κ is the restriction of μ to K . Indeed, in this case, there are equalities

$$\alpha(e \cdot \nu(f_{i+1}) + x_i) = \alpha(e_{p_i}), \quad 1 \leq i \leq m,$$

$$\alpha(e \cdot \nu(f_{i+1}) + x_i) = \alpha(x_i), \quad m + 1 \leq i \leq n.$$

Now we apply Remark 4.4 and formula (18).

Note that if $m < n$, then $\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1})) \equiv 0 \pmod{2}$, because $\nu(f_{n+1}) = 0$.

Remark 4.6. Example 4.5(ii) has the following generalization. Suppose that the $(n \times n)$ -matrix $(\nu(f_2), \dots, \nu(f_{n+1}))$ is upper-triangular with units on the diagonal. Then there is an equality

$$CC_n(f_1, f_2, \dots, f_{n+1}) = (-1)^{\text{sgn}(\nu(f_1), \dots, \nu(f_{n+1}))} \langle f_1, f_2, \dots, f_{n+1} \rangle.$$

Indeed, this follows from invariance of the map sgn under automorphisms of the $\mathbb{Z}(A)$ -module $\mathbb{Z}^n(A)$, formula (6), Corollary 3.2, and Example 4.5(ii) applied to the collection $\phi^{-1}(f_1), t_1, \dots, t_n \in \mathcal{L}^n(A)^*$, where $\phi: \mathcal{L}^n(A) \rightarrow \mathcal{L}^n(A)$ is a continuous automorphism such that $\phi(t_i) = f_{i+1}$, $1 \leq i \leq n$.

Remark 4.7. Here is a more explicit way to compute the higher-dimensional Contou-Carrère symbol, essentially using Example 4.5(iii), which is based on Theorem 4.3 and Remark 4.4. Actually, this gives also a correct definition (slightly different from formula (20)) of the higher-dimensional Contou-Carrère symbol over an arbitrary ring. Let A be any ring and let $f_1, \dots, f_{n+1} \in \mathcal{L}^n(A)^*$. In order to compute $CC_n(f_1, \dots, f_{n+1}) \in A^*$, one makes the following steps.

1. Take the decomposition into a finite product of rings $A \simeq \prod_{j=1}^N A_j$ such that for all j with $1 \leq j \leq N$, the restriction of $(\nu(f_1), \dots, \nu(f_{n+1}))$ to $\text{Spec}(A_j)$ is a constant \mathbb{Z}^{n^2} -valued function and for all j_1, j_2 with $1 \leq j_1 < j_2 \leq N$, the restrictions of $(\nu(f_1), \dots, \nu(f_{n+1}))$ to $\text{Spec}(A_{j_1})$ and $\text{Spec}(A_{j_2})$ are not equal. Clearly, this decomposition is unique. By functoriality of CC_n we have

$$CC_n(f_1, \dots, f_{n+1}) = \prod_{j=1}^N CC_n(f_{1,j}, \dots, f_{n+1,j}),$$

where $f_i = \prod_{j=1}^N f_{i,j}$ for any $1 \leq i \leq n+1$, and $f_{i,j} \in \mathcal{L}^n(A_j)^*$. Replace A by each of A_j , thus suppose from now on that all $\nu(f_i)$ are constant \mathbb{Z}^n -valued functions.

2. By decomposition (4) and multilinearity of CC_n , the symbol $CC_n(f_1, \dots, f_{n+1})$ decomposes uniquely into a product of symbols $CC_n(g_1, \dots, g_{n+1})$ such that for each i , $1 \leq i \leq n+1$, we have that either $g_i \in \{t_1, \dots, t_n\}$, or $\nu(g_i) = 0$. Replace the collection (f_1, \dots, f_{n+1}) by each collection of such type.
3. Suppose that there are equal elements among f_1, \dots, f_{n+1} and let $f_i = f_j$ be the equality with the smallest pair (i, j) , $i < j$, where we consider a natural lexicographical order on the pairs. Then put

$$CC_n(f_1, \dots, f_{n+1}) = (-1)^{\det(\nu(f_1), \dots, \widehat{\nu(f_j)}, \dots, \nu(f_{n+1}))},$$

where the notation $\widehat{\nu(f_j)}$ means that we omit $\nu(f_j)$. Note that here we use the antisymmetric property of CC_n and formula (11), which is true for any ring A , see reasonings in Section 2.

4. Suppose that all elements f_1, \dots, f_{n+1} are different. Then there is a unique permutation σ of the collection (f_2, \dots, f_{n+1}) such that σ preserves the order between f_i 's with $\nu(f_i) = 0$ and the result of the permutation is of type $(t_{p_1}, \dots, t_{p_m}, g_{m+1}, \dots, g_n)$, where $0 \leq m \leq n$, $1 \leq p_1 < \dots < p_m \leq n$, and $\nu(g_i) = 0$ for all i , $m+1 \leq i \leq n$. Using the antisymmetric property of CC_n , put

$$CC_n(f_1, f_2, \dots, f_{n+1}) = CC_n(f_1, t_{p_1}, \dots, t_{p_m}, g_{m+1}, \dots, g_n)^{\text{sgn}(\sigma)}$$

and replace the collection $(f_1, f_2, \dots, f_{n+1})$ by the collection $(f_1, t_{p_1}, \dots, t_{p_m}, g_{m+1}, \dots, g_n)$.

5. For a collection $(f_1, f_2, \dots, f_{n+1}) = (f_1, t_{p_1}, \dots, t_{p_m}, f_{m+2}, \dots, f_{n+1})$ such that $\nu(f_{m+1}) = \dots = \nu(f_{n+1}) = 0$, define $CC_n(f_1, f_2, \dots, f_{n+1})$ by formula (25).

Remark 4.8. Take positive integers $1 \leq j_1 < \dots < j_q \leq n$, where $1 \leq q \leq n$. Let $\mathbb{Q}[[x_{i,l}]]$ be the ring of power series in formal variables $x_{i,l}$, where $1 \leq i \leq n+1-p$ and $l \in \mathbb{Z}^n$, see [7, Def.5.1]. Consider infinite (in all "directions") series $f_i := 1 + \sum_{l \in \mathbb{Z}^n} x_{i,l} t^l$, where $1 \leq i \leq n+1-p$. Formally opening brackets in the expression (cf. formula (8))

$$\exp \operatorname{res} \left(\log(f_1) \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_p}{f_p} \wedge \frac{dt_{j_1}}{t_{j_1}} \wedge \dots \wedge \frac{dt_{j_q}}{t_{j_q}} \right),$$

we obtain a power series $\varphi_{n,j_1,\dots,j_q} \in \mathbb{Q}[[x_{i,l}]]$. A more rigorous definition of the series $\varphi_{n,j_1,\dots,j_q}$ is given in [7, §8.6], where it is also shown that, in fact, this series has integral coefficients. The proof of this fact uses the existence of the higher-dimensional Contou-Carrère symbol for all rings, which was constructed in [7] with the help of the boundary map for algebraic K -groups by formula (12), see the discussion before [7, Ex. 8.32]. Now, Theorem 4.3 gives an explicit formula for the higher-dimensional Contou-Carrère symbol for any ring. After all, this gives a new proof of the integrality of coefficients of the series $\varphi_{n,j_1,\dots,j_q}$, without the use of algebraic K -theory.

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